

# Formal Definition of the limit of a sequence: Re-Visited

Before the formal definition of the limit of a sequence is encountered, generally, a limit is described as a number to which the sequence gets closer and closer. In other words, later elements in the sequence are closer to the limit than the previous terms. This paper explores and makes this sense of getting closer and closer precise and also provides some interesting results.

**Definition 1.1** A sequence, denoted by  $(x_n)$ , in  $\mathbb{R}$  is called a non-constant sequence iff

$$\forall n \in \mathbb{N}, \exists k \in \mathbb{N} \text{ and } k > n \text{ s.t. } x_n \neq x_k$$

**Definition 1.2**  $\forall n \in \mathbb{N}, E_n = \{\epsilon > 0 : \forall m \geq n, m \in \mathbb{N} \implies |x_m - x| < \epsilon\}$

**Lemma 1.1** *if  $(x_n)$  is a convergent sequence, then  $\forall n \in \mathbb{N}, E_n \neq \phi$ .*

Let  $x_n$  converges to  $x \in \mathbb{R}$ . So,  $x_n$  is bounded i.e.  $|x_n| < M$ , for some  $M > 0$ . If  $n \in \mathbb{N}$ ,  $|x_n - x| \leq |x_n| + |x| < M + |x|$ . Thus,  $|x_n - x| < M + |x|$ ,  $\forall n \in \mathbb{N}$ . Hence,  $(M + |x|) \in E_n$ ,  $\forall n \in \mathbb{N}$ .

**Lemma 1.2** *For a non – constant, convergent sequence there exists  $\delta > 0$  s.t.  $\delta \notin E_n, \forall n \in \mathbb{N}$ . And,  $\delta < \epsilon$  for all  $\epsilon \in E_n$ .*

Assume for a non-constant, convergent sequence,  $(x_n)$  there exists  $E_n$  such that  $\epsilon \in E_n$  for all  $\epsilon > 0$ . So,  $0 \leq |x_m - x| < \epsilon$ ,  $m \geq n, m \in \mathbb{N}$ , for all  $\epsilon > 0 \implies |x_m - x| = 0 \implies x_m = x$  for all  $m \geq n$ . Thus,  $(x_n)$  is ultimately constant. This is a contradiction.

So, let  $n \in \mathbb{N}$ , and  $\delta > 0$  s.t.  $\delta \notin E_n$ . If  $\delta \geq \epsilon$  for some  $\epsilon \in E_n$  then  $|x_m - x| < \epsilon \leq \delta$ . Which implies  $\delta \in E_n$ . Thus,  $\delta < \epsilon$  for all  $\epsilon \in E_n$ .

**Lemma 1.3** *Let  $(x_n)$  be a non – constant, convergent sequence, then for every  $n \in \mathbb{N}$  there exists  $n_1, k \in \mathbb{N}$  and  $\delta_1 > 0$  with  $n \leq n_1 < k$  s.t.  $|x_{n_1} - x| \geq \delta_1$  but  $|x_r - x| < \delta_1$  for all  $r \geq k$ .*

Suppose,  $x_n$  is a non-constant sequence which converges to  $x$ . Then, from lemma 1.1 and 1.2 we can find  $\delta_n > 0$  such that  $\delta_n \notin E_n$  for all  $n \in \mathbb{N}$ . From Lemma 1.2,  $\delta_n < \epsilon$  for all  $\epsilon \in E_n$ . Since,  $(x_n)$  converges we can find  $k_n \in \mathbb{N}$  such that  $|x_m - x| < \delta_n$  for all  $m \geq k_n$ . Thus,  $\delta_n \in E_{k_n}$ .

Now, if  $M \leq n$  then if  $\delta \in E_M \implies |x_r - x| < \delta$  for all  $r \geq M$ . Since,  $M \leq n$  so,  $r \geq n \implies r \geq M$ . Hence,  $|x_r - x| < \delta$  for all  $r \geq n$ . Therefore,  $\delta \in E_n$  i.e.  $E_M$  is a subset of  $E_n$ .

Thus,  $k_n$  must be greater than  $n$ .

**Lemma 1.4** *Let  $(x_n)$  be a non-constant and converges to  $x \in \mathbb{R}$  Then, we can form a sequence  $(\delta_n)$  which converges to 0, such that there exists  $k_n$  with the following property for each  $n$ .*

1.  $n < k_n$
2.  $|x_r - x| < \delta_n$ , for all  $r \geq k_n$
3. There exists  $n' \geq n$  s.t.  $|x_{n'} - x| \geq \delta_n$

From lemma 1.1, 1.2 and 1.3, we can find  $\delta_1 > 0$  and  $n_1 \in \mathbb{N}$  s.t. above three conditions holds for  $n = 1$ . In the same way we can find  $\delta_{n_1} > 0$  and  $n_{n_1}$  s.t. above three conditions holds for  $n = n_1$ . Continuing in this way we can form a strictly decreasing sequence lets denote it by  $(\delta_n)$ . Now, since  $(x_n)$  converges to  $x$ , for each  $\epsilon > 0$  there exists  $E_p$  s.t.  $\epsilon \in E_p$ . Also,  $(x_n)$  is non-constant so, there exists  $\epsilon'$  s.t.  $\epsilon > \epsilon'$ .

Here,  $(\delta_n)$  converges as it is bounded and strictly decreasing. Lets, denote the limit of  $(\delta_n)$  as  $\delta$ . Hence, from all the preceding arguments we have  $0 \leq \delta < \epsilon$  for all  $\epsilon > 0$ . Hence  $\delta = 0$ .

Until now, we have derived some basic and interesting consequences of the definition of the limit.

So, conversely assume that: *Let  $(x_n)$  be a non-constant and we can form a positive sequence  $(\delta_n)$  which converges to 0, such that there exists  $k_n$  with the following property for each  $n$ .*

1.  $n < k_n$
  2.  $|x_r - x| < \delta_n$ , for all  $r \geq k_n$
  3. There exists  $n' \geq n$  s.t.  $|x_{n'} - x| \geq \delta_n$
- $\delta_{n+1} < \delta_n, \forall n \in \mathbb{N}$ .

Now, if  $\epsilon > 0$  is given there exists  $m \in \mathbb{N}$  s.t.  $\delta_m < \epsilon$  because the sequence  $(\delta_n)$  converges to 0. Thus,  $|x_r - x| < \epsilon$  for all  $r \geq k_m$ . As  $|x_r - x| < \delta_m, \forall r \geq k_m$  Hence, the sequence  $(x_n)$  converges to  $x \in \mathbb{R}$ .